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INVARIANT INTEGRALS IN THE PROBLEM OF A CRACK ON THE INTERFACE BETWEEN TWO MEDIA

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The equilibrium problem for an elastic body containing a crack on the interface between two media is considered. It is proved that there exist invariant (independent of the integration surface) integrals in this problem. The existence of invariant integrals is also established in the problem of a contact between an elastic body and a rigid stamp. Nonlinear boundary conditions of mutual non-penetration are prescribed on the contact boundaries. The physical meaning of invariant integrals is established.

Key words: *invariant integral, elastic body, crack, contact problem.*

Introduction. A contact problem describes an equilibrium state for an elastic body contacting a rigid (nondeformable) body. Boundary conditions of equality and inequality types are prescribed on the contact boundary. In the equilibrium problem for an elastic body containing a crack, nonlinear conditions on the crack faces are also prescribed. In the present work, we prove that there exist invariant integrals in these nonlinear problems. Invariant integrals are constructed both in the two-dimensional and in the three-dimensional cases.

The existence of invariant integrals in the linear crack theory, which are commonly called the Cherepanov–Rice integrals, was discussed in many papers (see, e.g., [1–4]). These discussions involved linear problems, which means setting linear boundary conditions on the crack faces. We will consider nonlinear problems of the crack theory, which were analyzed in [5]. The specific features of nonlinear problems are the boundary conditions on the crack faces, which have the form of a system of equalities and inequalities. From the viewpoint of applications, nonlinear problems provide a better description of real processes, whereas linear problems of the crack theory can contradict the mechanics of the phenomenon. Invariant integrals for smooth (in particular, constant) tensors of elasticity moduli were constructed previously in nonlinear crack problems [5–7]. In the present work, we construct invariant integrals for an elastic body with a crack located on the interface between two media. In this case, the tensor of elasticity moduli is not smooth in the domain.

To obtain invariant integrals in contact problems, a fictitious domain method is used, which was recently developed for problems with Signorini boundary conditions [8, 9]. In this case, the equilibrium problem for a cracked body belongs to a family of parameter-dependent problems, and the contact problem corresponds to the limiting value of the parameter. Actually, invariant integrals in the problems considered, i.e., in the problem of equilibrium of an anisotropic body with a crack and in the contact problem, would be obtained simultaneously. The fictitious domain method used allows us, by introducing an auxiliary parameter, to construct a family of boundary-value problems including both the contact problem and the equilibrium problem for a cracked body. Fundamentals of the fictitious domain method, as applied to linear boundary conditions, can be found in [10–12]. Simultaneously, we use a formula for the derivative of the energy functional with respect to the perturbation parameter in problems of the elasticity theory for cracked bodies with nonlinear boundary conditions on the crack faces. The technique of differentiation of energy functionals in nonlinear crack problems is described in [5–7, 13, 14]. Applications of crack problems in solid mechanics can be found in [1, 2, 15], and the global issues of studying boundary-value problems in non-smooth domains were considered in [16].

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Two-Dimensional Case. Let $\Omega_1 \subset \mathbb{R}^2$ be a simply connected bounded domain with the Lipschitz boundary Γ_1, and $\Gamma_c \subset \Gamma_1$ be a contact boundary, which is assumed, for simplicity, to be a smooth curve defined in the form of a graph of the function $x_2 = \phi(x_1)$, $x_1 \in [0,1]$. It is assumed that there exists $\delta_0 > 0$ such that

$$((-\delta_0, \delta_0) \times \{0\}) \subset \Gamma_1, \qquad (1-\delta_0, 1+\delta_0) \times \{0\}) \subset \Gamma_1. \tag{1}$$

These inclusions mean that the boundary Γ_1 contains straight-line segments in the neighborhood of the points $(0,0)$ and $(1,0)$. We denote the unit vector of the internal normal to Γ_1 by $\boldsymbol{\nu} = (\nu_1, \nu_2)$. Let $\Gamma_0 = \Gamma_1 \setminus \Gamma_c$. The contact problem is formulated as follows [17]. In the domain Ω_1, we have to find functions $\boldsymbol{u}^0 = (u_1^0, u_2^0)$ and $\sigma = \{\sigma_{ij}\}$ $(i,j = 1,2)$ such that

$$-\operatorname{div} \sigma = \boldsymbol{f} \qquad \text{in} \quad \Omega_1; \tag{2}$$

$$\sigma = C^1 \varepsilon(\boldsymbol{u}^0) \qquad \text{in} \quad \Omega_1; \tag{3}$$

$$\boldsymbol{u}^0 = 0 \qquad \text{on} \quad \Gamma_0; \tag{4}$$

$$\boldsymbol{u}^0 \cdot \boldsymbol{\nu} \geqslant 0, \quad \sigma_\nu \leqslant 0, \quad \boldsymbol{\sigma}_\tau = 0, \quad \boldsymbol{u}^0 \cdot \boldsymbol{\nu} \sigma_\nu = 0 \qquad \text{on} \quad \Gamma_c. \tag{5}$$

Hereinafter, $\varepsilon_{ij}(\boldsymbol{v}) = (v_{i,j} + v_{j,i})/2$ are the components of the strain tensor, $v_{i,j} = \partial v_i / \partial x_j$, $x = (x_1, x_2) \in \Omega_1$, $\boldsymbol{f} = (f_1, f_2) \in C^1_{\text{loc}}(\mathbb{R}^2)$ is a known function, $C^1 = \{c^1_{ijkl}\}$ is the tensor of elasticity moduli $(i,j,k,l = 1,2)$,

$$c^1_{ijkl} = c^1_{klij} = c^1_{jikl}, \qquad c^1_{ijkl} = \text{const},$$

$$c^1_{ijkl} \xi_{kl} \xi_{ij} \geqslant c|\xi|^2, \qquad c > 0 \quad \forall \;\; \xi = \{\xi_{ij}\}, \tag{6}$$

$$\sigma_\nu = \sigma_{ij} \nu_j \nu_i, \qquad \boldsymbol{\sigma}_\tau = \sigma \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}, \qquad \sigma \boldsymbol{\nu} = \{\sigma_{ij} \nu_j\}_{i=1}^2.$$

In this system, Eqs. (2) are the equilibrium equations, Eqs. (3) describe Hooke's law, the boundary condition (4) corresponds to clamping of the elastic body on Γ_0, and the boundary conditions (5) describe the contact of the elastic body with a non-deformable surface with zero friction and are called the Signorini boundary conditions. All quantities with two subscripts are assumed to be symmetric with respect to these subscripts ($\sigma_{ij} = \sigma_{ji}$, etc.); summation is performed over repeated subscripts.

It is known that problem (2)–(5) admits a variational formulation and has a unique solution. Indeed, let us consider the space of the Sobolev functions

$$H^1_{\Gamma_0}(\Omega_1) = \{\boldsymbol{v} = (v_1, v_2) \in H^1(\Omega_1) \mid \boldsymbol{v} = 0 \text{ on } \Gamma_0\}$$

and the set of admissible displacements

$$K = \{\boldsymbol{v} \in H^1_{\Gamma_0}(\Omega_1) \mid \boldsymbol{v} \cdot \boldsymbol{\nu} \geqslant 0 \text{ a. e. on } \Gamma_c\}.$$

Then, problem (2)–(5) is equivalent to minimization of the functional

$$\Pi_0(\Omega_1; \boldsymbol{v}) = \frac{1}{2} \int\limits_{\Omega_1} \sigma(\boldsymbol{v}) \varepsilon(\boldsymbol{v}) - \int\limits_{\Omega_1} \boldsymbol{f} \boldsymbol{v}$$

over the set K and can be written in the form of the variational inequality

$$\boldsymbol{u}^0 \in K, \qquad \int\limits_{\Omega_1} \sigma(\boldsymbol{u}^0) \varepsilon(\boldsymbol{v} - \boldsymbol{u}^0) \geqslant \int\limits_{\Omega_1} \boldsymbol{f}(\boldsymbol{v} - \boldsymbol{u}^0) \qquad \forall \; \boldsymbol{v} \in K. \tag{7}$$

Hereinafter, we have $\sigma(\boldsymbol{v}) = C^1 \varepsilon(\boldsymbol{v})$.

In addition to the contact problem (2)–(5), we consider the equilibrium problem for an elastic body with a crack on the interface between two media. Adding a bounded domain Ω_2 with the Lipschitz boundary Γ_2 to the domain Ω_1 and solving the boundary-value problem with nonlinear boundary conditions on Γ_c in the domain $\Omega_c = \Omega_1 \cup \Omega_2 \cup (\Sigma \setminus \Gamma_c)$, we can establish the existence of invariant integrals in the equilibrium problem for an anisotropic elastic body with a crack on the interface between the media. Here, we have $\Sigma = \Sigma_0 \setminus \partial \Sigma_0$ and $\Sigma_0 = \Gamma_1 \cap \Gamma_2$. The resultant problem describes equilibrium of an elastic body occupying the domain Ω_c and containing the crack Γ_c, with the boundary conditions of non-penetration on the faces $\Gamma_c^\pm$. Actually, we consider a family of boundary-value problems depending on the parameter λ. Each value of the parameter $\lambda > 0$ corresponds to the equilibrium problem for a cracked body, $\lambda = 0$ corresponds to problem (2)–(5). The existence of invariant integrals will be established simultaneously for the entire family of problems, i.e., for all $\lambda > 0$. Passing to the limit as $\lambda \to 0$, we also establish the existence of invariant integrals for the contact problem (2)–(5).

For the contact problem (2)–(5), the added domain Ω_2 is called fictitious. As will be shown below, the coefficients of the operator for the problem considered in the domain Ω_2 tend to infinity as λ tends to zero.

Before implementing the scheme described above, we clarify the geometry of the domains Ω_1 and Ω_2. We assume that the points $(0,0)$ and $(1,0)$ are internal points of the curve Σ (this assumption does not refer to examples 3 and 4, where another geometry of the domains is considered). Concerning the smoothness of the boundaries Γ_1 and Γ_2, it is sufficient to satisfy the Lipschitz condition. Note that the existence of invariant integrals of different types and for domains of different geometry will be established. Each two-dimensional case requires integration over an (arbitrary) smooth curve; in three-dimensional cases, integration is performed over two-dimensional surfaces.

Thus, we introduce the tensor $B^\lambda = \{b^\lambda_{ijkl}\}$ $(\lambda > 0,\ i,j,k,l = 1,2)$,

$$b^\lambda_{ijkl} = \begin{cases} c^1_{ijkl} & \text{in } \Omega_1, \\ \lambda^{-1} c^2_{ijkl} & \text{in } \Omega_2. \end{cases}$$

Here, the tensor $C^2 = \{c^2_{ijkl}\}$ possesses the same properties as the tensor C^1. In the domain Ω_c containing the crack Γ_c, we solve the following problem. We have to find functions $\boldsymbol{u}^\lambda = (u^\lambda_1, u^\lambda_2)$ and $\sigma^\lambda = \{\sigma^\lambda_{ij}\}$ $(i,j = 1,2)$ such that

$$-\operatorname{div} \sigma^\lambda = \boldsymbol{f} \qquad \text{in} \quad \Omega_c; \tag{8}$$

$$\sigma^\lambda = B^\lambda \varepsilon(\boldsymbol{u}^\lambda) \qquad \text{in} \quad \Omega_c; \tag{9}$$

$$\boldsymbol{u}^\lambda = 0 \qquad \text{on} \quad \Gamma; \tag{10}$$

$$[\boldsymbol{u}^\lambda] \cdot \boldsymbol{\nu} \geqslant 0, \quad [\sigma^\lambda_\nu] = 0, \quad \sigma^\lambda_\nu \leqslant 0, \quad \boldsymbol{\sigma}^\lambda_\tau = 0, \quad [\boldsymbol{u}^\lambda] \cdot \boldsymbol{\nu} \sigma^\lambda_\nu = 0 \quad \text{on} \quad \Gamma_c. \tag{11}$$

Here $[\boldsymbol{v}] = \boldsymbol{v}^+ - \boldsymbol{v}^-$ is the jump of the function $\boldsymbol{v}$ on Γ_c (the plus and minus refer to the positive and negative directions of the normal $\boldsymbol{\nu}$, respectively), Γ is the outer boundary of the domain Ω_c, i.e., $\Gamma = \partial\Omega_c \setminus (\Gamma^+_c \cup \Gamma^-_c)$, $\sigma^\lambda_\nu = \sigma^\lambda_{ij}\nu_j\nu_i$, and $\boldsymbol{\sigma}^\lambda_\tau = \sigma^\lambda \boldsymbol{\nu} - \sigma^\lambda_\nu \boldsymbol{\nu}$. The equality $\boldsymbol{\sigma}^\lambda_\tau = 0$ on Γ_c means that $\boldsymbol{\sigma}^\lambda_\tau = 0$ on $\Gamma^\pm_c$.

Each value of the parameter $\lambda > 0$ corresponds to the equilibrium problem for a body with a crack on the interface between anisotropic parts that occupy the domains Ω_1 and Ω_2 with constant elasticity tensors C^1 and C^2/λ, respectively. Let us consider the case $\lambda > 0$ and the limiting case $\lambda = 0$.

Problem (8)–(11) has a unique solution for each particular $\lambda > 0$. Indeed, let us consider the space of the functions

$$H^1_\Gamma(\Omega_c) = \{\boldsymbol{v} = (v_1, v_2) \in H^1(\Omega_c) \mid \boldsymbol{v} = 0 \text{ on } \Gamma\}$$

and the set of admissible displacements

$$K_c = \{\boldsymbol{v} \in H^1_\Gamma(\Omega_c) \mid [\boldsymbol{v}] \cdot \boldsymbol{\nu} \geqslant 0 \text{ a. e. on } \Gamma_c\}.$$

Then, problem (8)–(11) is equivalent to minimization of the functional

$$\Pi_\lambda(\Omega_c; \boldsymbol{v}) = \frac{1}{2} \int\limits_{\Omega_c} \sigma^\lambda(\boldsymbol{v}) \varepsilon(\boldsymbol{v}) - \int\limits_{\Omega_c} \boldsymbol{f}\boldsymbol{v}$$

over the set K_c and can be formulated in the form of the variational inequality

$$\boldsymbol{u}^\lambda \in K_c, \qquad \int\limits_{\Omega_c} \sigma^\lambda(\boldsymbol{u}^\lambda) \varepsilon(\boldsymbol{v} - \boldsymbol{u}^\lambda) \geqslant \int\limits_{\Omega_c} \boldsymbol{f}(\boldsymbol{v} - \boldsymbol{u}^\lambda) \qquad \forall\, \boldsymbol{v} \in K_c. \tag{12}$$

Here $\sigma^\lambda(\boldsymbol{v})$ are found from the equation of the form (9), i.e., $\sigma^\lambda(\boldsymbol{v}) = B^\lambda \varepsilon(\boldsymbol{v})$.

The objective of further considerations is to introduce a perturbation parameter into problem (12), i.e., to consider a family of perturbed problems depending on the parameter δ and defined in the perturbed domain Ω^δ_c. For each fixed λ and small δ, we will find the solution of the perturbed problem $\boldsymbol{u}^{\lambda\delta}$ and the derivative of the energy functional $\Pi_\lambda(\Omega_c^\delta; \boldsymbol{u}^{\lambda\delta})$ with respect to the parameter δ for $\delta = 0$. With a proper choice of perturbations, the formula for the derivative will yield invariant integrals in problem (8)–(11). Then, we pass to the limit in the formula for this derivative as $\lambda \to 0$. It is important to note that the formula for the above-mentioned derivative of the energy functional will contain the solution $\boldsymbol{u}^\lambda$, which is unperturbed with respect to δ. In addition, $\boldsymbol{u}^\lambda$ will converge to $\boldsymbol{u}^0$ as $\lambda \to 0$, where $\boldsymbol{u}^0$ is the solution of problem (7), which allows us to pass to the limit as $\lambda \to 0$ in the formula for the derivative mentioned. The final formula leads to invariant integrals for problem (2)–(5) with an appropriate choice of perturbations.

We consider the perturbation of the domain Ω_c and seek for the solution of the problem in the perturbed domain Ω_c^δ. Let the transformation of independent variables

$$y = \Psi_\delta(x), \qquad x \in \Omega_c, \quad y \in \Omega_c^\delta \tag{13}$$

describe the perturbation of the domain Ω_c, where $\Psi_\delta(x) = x + \delta \boldsymbol{V}(x)$; $\boldsymbol{V}(x) = (V_1(x), V_2(x)) \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2)$. For small δ, transformation (13) establishes a biunique correspondence between Ω_c and Ω_c^δ. We assume that the vector field $\boldsymbol{V}(x)$ is such that

$$\boldsymbol{\nu}^\delta(y) = \boldsymbol{\nu}(x), \qquad y = \Psi_\delta(x), \tag{14}$$

where $\boldsymbol{\nu}^\delta(y)$ is the normal to the perturbed cut $\Gamma_c^\delta = \Psi_\delta(\Gamma_c)$. For each δ, we obtain a perturbed domain Ω_c^δ and a perturbed [as compared to (8)–(11)] boundary-value problem, which is formulated as follows. We have to find functions $\boldsymbol{u}^{\lambda\delta} = (u_1^{\lambda\delta}, u_2^{\lambda\delta})$ and $\sigma^{\lambda\delta} = \{\sigma_{ij}^{\lambda\delta}\}$ $(i, j = 1, 2)$ such that

$$-\operatorname{div} \sigma^{\lambda\delta} = \boldsymbol{f} \qquad \text{in} \quad \Omega_c^\delta; \tag{15}$$

$$\sigma^{\lambda\delta} = B^{\lambda\delta} \varepsilon(\boldsymbol{u}^{\lambda\delta}) \qquad \text{in} \quad \Omega_c^\delta; \tag{16}$$

$$\boldsymbol{u}^{\lambda\delta} = 0 \qquad \text{on} \quad \Psi_\delta(\Gamma); \tag{17}$$

$$[\boldsymbol{u}^{\lambda\delta}] \cdot \boldsymbol{\nu} \geqslant 0, \quad [\sigma_\nu^{\lambda\delta}] = 0, \quad \sigma_\nu^{\lambda\delta} \leqslant 0, \quad \boldsymbol{\sigma}_\tau^{\lambda\delta} = 0, \quad [\boldsymbol{u}^{\lambda\delta}] \cdot \boldsymbol{\nu} \sigma_\nu^{\lambda\delta} = 0 \qquad \text{on} \quad \Gamma_c^\delta. \tag{18}$$

We assume that the coefficients $b_{ijkl}^{\lambda\delta}$ in Eq. (16) are determined in Ω_c^δ with properties of smoothness being preserved during transformation (13), i.e., they remain piecewise-constant:

$$b_{ijkl}^{\lambda\delta} = \begin{cases} c_{ijkl}^1 & \text{on } \Psi_\delta(\Omega_1), \\ \lambda^{-1} c_{ijkl}^2 & \text{on } \Psi_\delta(\Omega_2). \end{cases}$$

Let $\boldsymbol{u}^{\lambda\delta}$ be the solution of problem (15)–(18) from the space $H^1(\Omega_c^\delta)$. This solution can be found by the following procedure. We consider the set of admissible displacements in problem (15)–(18):

$$K_c^\delta = \{\boldsymbol{v} \in H^1_{\Psi_\delta(\Gamma)}(\Omega_c^\delta) \mid [\boldsymbol{v}] \cdot \boldsymbol{\nu} \geqslant 0 \text{ a. e. on } \Gamma_c^\delta\}.$$

Next, we introduce the notation

$$\Pi_\lambda(\Omega_c^\delta; \boldsymbol{v}) = \frac{1}{2} \int\limits_{\Omega_c^\delta} \sigma^{\lambda\delta}(\boldsymbol{v}) \varepsilon(\boldsymbol{v}) - \int\limits_{\Omega_c^\delta} \boldsymbol{f} \boldsymbol{v}$$

and consider the minimization problem

$$\min_{v \in K_c^\delta} \Pi_\lambda(\Omega_c^\delta; \boldsymbol{v}). \tag{19}$$

The solution of problem (19) exists and is determined from the variational inequality

$$\boldsymbol{u}^{\lambda\delta} \in K_c^\delta, \quad \int\limits_{\Omega_c^\delta} \sigma^{\lambda\delta}(\boldsymbol{u}^{\lambda\delta}) \varepsilon(\boldsymbol{v} - \boldsymbol{u}^{\lambda\delta}) \geqslant \int\limits_{\Omega_c^\delta} \boldsymbol{f}(\boldsymbol{v} - \boldsymbol{u}^{\lambda\delta}) \quad \forall\, \boldsymbol{v} \in K_c^\delta. \tag{20}$$

We assume that $\boldsymbol{V}(x) = (V_1(x), 0)$, and the function V_1 is such that $\Psi_\delta(\Gamma) = \Gamma$ and condition (14) is satisfied. In this case, mapping (13) establishes a biunique correspondence between the spaces $H^1_\Gamma(\Omega_c)$ and $H^1_\Gamma(\Omega_c^\delta)$, and also between the sets K_c and K_c^δ. Let us determine the energy functional in problem (20) as

$$\Pi_\lambda(\Omega_c^\delta; \boldsymbol{u}^{\lambda\delta}) = \frac{1}{2} \int\limits_{\Omega_c^\delta} \sigma^{\lambda\delta}(\boldsymbol{u}^{\lambda\delta})\varepsilon(\boldsymbol{u}^{\lambda\delta}) - \int\limits_{\Omega_c^\delta} \boldsymbol{f}\boldsymbol{u}^{\lambda\delta}$$

and introduce the notation

$$I^\lambda = \frac{d}{d\delta} \Pi_\lambda(\Omega_c^\delta; \boldsymbol{u}^{\lambda\delta})\Big|_{\delta=0}$$

for the derivative of the energy functional with respect to the parameter δ. According to [6, 7], we have

$$I^\lambda = \int\limits_{\Omega_c} \Big\{ \frac{1}{2} \operatorname{div}(\boldsymbol{V} b_{ijkl}^\lambda)\varepsilon_{kl}(\boldsymbol{u}^\lambda)\varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma_{ij}^\lambda(\boldsymbol{u}^\lambda) E_{ij}\Big(\frac{\partial \boldsymbol{V}}{\partial x}; \boldsymbol{u}^\lambda\Big)\Big\} - \int\limits_{\Omega_c} \operatorname{div}(\boldsymbol{V} f_i) u_i^\lambda. \tag{21}$$

Here, $E_{ij}(\Phi; \boldsymbol{v}) = (v_{i,k}\Phi_{kj} + v_{j,k}\Phi_{ki})/2$ and $\Phi = \{\Phi_{ij}\}$ $(i, j = 1, 2)$. Note, by virtue of the assumption made about the vector field $\boldsymbol{V}$, there is no need to differentiate the coefficients b_{ijkl}^λ with respect to x_2, which, generally speaking, have a discontinuity along the curve Σ. We write the formula for I^λ in the form $I^\lambda = I_1^\lambda + I_2^\lambda$, where

$$\begin{aligned} I_1^\lambda &= \int\limits_{\Omega_1} \Big\{ \frac{1}{2} \operatorname{div} \boldsymbol{V} \sigma_{ij}(\boldsymbol{u}^\lambda)\varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma_{ij}(\boldsymbol{u}^\lambda) E_{ij}\Big(\frac{\partial \boldsymbol{V}}{\partial x}; \boldsymbol{u}^\lambda\Big)\Big\} - \int\limits_{\Omega_1} \operatorname{div}(\boldsymbol{V} f_i) u_i^\lambda, \\ I_2^\lambda &= \int\limits_{\Omega_2} \Big\{ \frac{1}{2} \operatorname{div} \boldsymbol{V} \sigma_{ij}^\lambda(\boldsymbol{u}^\lambda)\varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma_{ij}^\lambda(\boldsymbol{u}^\lambda) E_{ij}\Big(\frac{\partial \boldsymbol{V}}{\partial x}; \boldsymbol{u}^\lambda\Big)\Big\} - \int\limits_{\Omega_2} \operatorname{div}(\boldsymbol{V} f_i) u_i^\lambda. \end{aligned} \tag{22}$$

It is known (see [8, 9]) that, as $\lambda \to 0$,

$$\boldsymbol{u}^\lambda/\sqrt{\lambda} \to 0 \qquad \text{strongly in} \quad H^1(\Omega_2); \tag{23}$$

$$\boldsymbol{u}^\lambda \to \boldsymbol{u}^0 \qquad \text{strongly in} \quad H^1(\Omega_1), \tag{24}$$

where $\boldsymbol{u}^0$ is the solution of problem (2)–(5) (or problem (7)). It follows from (23) that

$$|\nabla \boldsymbol{u}^\lambda|^2/\lambda \to 0 \qquad \text{strongly in} \quad L^1(\Omega_2), \quad \lambda \to 0. \tag{25}$$

Then, from (21) with allowance for (22), (24), and (25), we find $I^0 = \lim_{\lambda\to 0} I^\lambda$, i.e., we have

$$I^0 = \int\limits_{\Omega_1} \Big\{ \frac{1}{2} \operatorname{div} \boldsymbol{V} \sigma_{ij}(\boldsymbol{u}^0)\varepsilon_{ij}(\boldsymbol{u}^0) - \sigma_{ij}(\boldsymbol{u}^0) E_{ij}\Big(\frac{\partial \boldsymbol{V}}{\partial x}; \boldsymbol{u}^0\Big)\Big\} - \int\limits_{\Omega_1} \operatorname{div}(\boldsymbol{V} f_i) u_i^0. \tag{26}$$

Note, the function $\boldsymbol{u}^0$ in Eq. (26) is the solution of problem (2)–(5).

The invariant integrals in problems (2)–(5) and (8)–(11) will be obtained from formulas (26) and (21), respectively. As the components of the stress tensor are not determined, generally speaking, in the domain Ω_2 for $\lambda = 0$, the corresponding invariant integrals for problems (2)–(5) and (8)–(11) will be written separately.

Let us now consider some particular cases of choosing the vector field $\boldsymbol{V}$, which will yield invariant integrals by means of transformations of formulas (21) and (26). In all examples, we will have to choose the neighborhoods S_1 and S_2 with smooth (Lipschitz) boundaries ∂S_1 and ∂S_2. In what follows, we assume that the boundaries of the domains $(S_1 \setminus S_2) \cap \Omega_c$ also satisfy the Lipschitz condition.

Example 1. Let the carrier of the function θ lie in a small neighborhood S_1 of the point $(1, 0)$ and $\theta = 1$ in the neighborhood S_2 of the point $(1, 0)$, $S_2 \subset S_1$. The smallness of the neighborhood S_1 means that ∂S_1 intersects the axis x_1 along straight-line segments (1). We choose perturbation (13) in the form

$$y_1 = x_1 + \delta\theta(x_1, x_2), \qquad y_2 = x_2,$$

where $(x_1, x_2) \in \Omega_c$ and $(y_1, y_2) \in \Omega_c^\delta$. The vector field $\boldsymbol{V}(x)$ is determined by the formula $\boldsymbol{V}(x) = (\theta(x), 0)$, and Eq. (21) can be rewritten as

$$I^\lambda = \int\limits_{\Omega_c} \Big\{ \frac{1}{2} \theta_{,1} \sigma_{ij}^\lambda(\boldsymbol{u}^\lambda)\varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma_{ij}^\lambda(\boldsymbol{u}^\lambda) u_{i,1}^\lambda \theta_{,j}\Big\} - \int\limits_{\Omega_c} (\theta f_i)_{,1} u_i^\lambda. \tag{27}$$

After integration by parts, Eq. (27) yields

$$I^\lambda = \int\limits_{(\partial S_2)\cap\overline{\Omega}_c} \Big\{\frac{1}{2}\, n_1 \sigma^\lambda_{ij}(\boldsymbol{u}^\lambda)\varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma^\lambda_{ij}(\boldsymbol{u}^\lambda) u^\lambda_{i,1} n_j\Big\} + \int\limits_{(S_1\backslash S_2)\cap\Omega_c} \theta(\sigma^\lambda_{ij,j} + f_i) u^\lambda_{i,1} + \int\limits_{S_2\cap\Omega_c} f_i u^\lambda_{i,1}. \tag{28}$$

Here $\boldsymbol{n} = (n_1, n_2)$ is the internal normal to the boundary ∂S_2, and $(\partial S_2)\cap\overline{\Omega}_c$ is a closed curve surrounding the crack tip $(1,0)$. It should be emphasized that the solution $\boldsymbol{u}^\lambda$ of problem (8)–(11) is H^2-smooth up to the points $(1-\delta_0, 1)\times\{0\}$ and $(1, 1+\delta_0)\times\{0\}$ (see [5, p. 100]), which ensures convergence of integrals in Eq. (28). In addition, it should be noted that integration in Eq. (28) can be performed with respect to either crack face if the part of the curve $(\partial S_2)\cap\overline{\Omega}_c$ lies on the segment $(1-\delta_0, 1)\times\{0\}$. This is valid due to the presence of the boundary conditions

$$\sigma^\lambda_{12}(\boldsymbol{u}^\lambda) = [\sigma^\lambda_{22}(\boldsymbol{u}^\lambda)] = 0, \qquad \sigma^\lambda_{22}(\boldsymbol{u}^\lambda)[u^\lambda_{2,1}] = 0 \quad \text{on} \quad (1-\delta_0, 1)\times\{0\}. \tag{29}$$

Indeed, the conditions $\sigma^\lambda_{12}(\boldsymbol{u}^\lambda) = 0$ and $[\sigma^\lambda_{22}(\boldsymbol{u}^\lambda)] = 0$ on $(1-\delta_0, 1)\times\{0\}$ coincide with the conditions $\boldsymbol{\sigma}^\lambda_\tau = 0$ and $[\sigma^\lambda_\nu] = 0$ [see (11)], and the proof of the second relation in (29) can be found in [5, p. 276].

We assume that $\boldsymbol{f} \equiv 0$ in $S_2\cap\Omega_c$. Taking into account the validity of the equilibrium equations (8) in Ω_c, we obtain the invariant integral for problem (8)–(11) from Eq. (28):

$$I^\lambda = \int\limits_{(\partial S_2)\cap\overline{\Omega}_c} \Big\{\frac{1}{2}\, n_1 \sigma^\lambda_{ij}(\boldsymbol{u}^\lambda)\varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma^\lambda_{ij}(\boldsymbol{u}^\lambda) u^\lambda_{i,1} n_j\Big\},$$

which is independent of the choice of the curve $(\partial S_2)\cap\overline{\Omega}_c$. By similar considerations, under the same conditions on $\boldsymbol{f}$, we obtain the invariant integral for problem (2)–(5) from Eq. (26):

$$I^0 = \int\limits_{(\partial S_2)\cap\Omega_1} \Big\{\frac{1}{2}\, n_1 \sigma_{ij}(\boldsymbol{u}^0)\varepsilon_{ij}(\boldsymbol{u}^0) - \sigma_{ij}(\boldsymbol{u}^0) u^0_{i,1} n_j\Big\}. \tag{30}$$

In this case, the curve $(\partial S_2)\cap\Omega_1$ is an arbitrary "cap" lying in Ω_1 and surrounding the point $(1,0)$.

In deriving Eq. (30) from (26), we should note the validity of the boundary condition

$$\sigma_{22}(\boldsymbol{u}^0) u^0_{2,1} = 0 \quad \text{on} \quad (1-\delta_0, 1)\times\{0\}, \tag{31}$$

and also the H^2-smoothness of the solution $\boldsymbol{u}^0$ up to the points $(1-\delta_0, 1)\times\{0\}$. This smoothness of the solution $\boldsymbol{u}^0$ of the contact problem (2)–(5) was proved in [17], and the validity of the boundary condition (31) can be established similar to the second relation in (29).

The invariant integral over the curve lying in Ω_1 and surrounding the point $(0,0)$ also exists and has the form (30).

Example 2. Let θ be a smooth function with a support in a small neighborhood S_1 of the curve Γ_c. Moreover, $\theta = 1$ in the neighborhood S_2 of the curve Γ_c, $S_2 \subset S_1$. We consider perturbation (13) in the form

$$y_1 = x_1 + \delta\theta(x), \qquad y_2 = x_2,$$

where $(x_1, x_2) \in \Omega_c$ and $(y_1, y_2) \in \Omega^\delta_c$. As in example 1, we have $\boldsymbol{V}(x) = (\theta(x), 0)$, and formula (21) coincides with (27).

Assuming that $\boldsymbol{f} \equiv 0$ in $S_2\cap\Omega_c$, we perform integration by parts in (27). We obtain the invariant integral for problem (8)–(11)

$$I^\lambda = \int\limits_{(\partial S_2)\cap\overline{\Omega}_c} \Big\{\frac{1}{2}\, n_1 \sigma_{ij}(\boldsymbol{u}^\lambda)\varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma_{ij}(\boldsymbol{u}^\lambda) u^\lambda_{i,1} n_j\Big\},$$

where $\boldsymbol{n} = (n_1, n_2)$ is the internal normal to ∂S_2. In this case, $(\partial S_2)\cap\overline{\Omega}_c$ is a curve lying in $\overline{\Omega}_c$ and surrounding Γ_c.

For problem (2)–(5), the invariant integral is obtained with the same choice of $\boldsymbol{V}(x)$ in (26) and has the form

$$I^0 = \int\limits_{(\partial S_2)\cap\Omega_1} \Big\{\frac{1}{2}\, n_1 \sigma_{ij}(\boldsymbol{u}^0)\varepsilon_{ij}(\boldsymbol{u}^0) - \sigma_{ij}(\boldsymbol{u}^0) u^0_{i,1} n_j\Big\}.$$

Now let us consider another geometry of the domains Ω_1 and Ω_2.

Example 3. Let a bounded domain Ω_1 have the form of a stripe. We assume that Ω_1 has a boundary consisting of segments Γ_0 and Γ_c of the form

$$\Gamma_0 = ((0,1) \times \{0\}) \cup ((0,1) \times \{1\}) \cup (\{0\} \times [0,1]),$$

$$\Gamma_c = \{(x_1, x_2) \mid x_1 = \psi(x_2),\ x_2 \in [0,1]\}.$$

We assume that the function ψ satisfies the Lipschitz condition; $0 < \psi(x_2) < 2$, where$x_2 \in [0,1]$. The domain Ω_2 also has the form of a bounded stripe with the boundary

$$\Gamma_2 = \Gamma_c \cup ((1,2) \times \{0\}) \cup ((1,2) \times \{1\}) \cup (\{2\} \times [0,1]).$$

Let the smooth function θ vanish outside a certain neighborhood S_1 of the curve Γ_c and there exists a neighborhood S_2 of the curve Γ_c, where $\theta = 1$ and $S_2 \subset S_1$. We consider the transformation $y = \Psi_\delta(x)$ of the form

$$y_1 = x_1 + \delta\theta(x), \qquad y_2 = x_2.$$

Here $(x_1, x_2) \in \Omega_c$ and $(y_1, y_2) \in \Omega_c^\delta$. As previously, $\Omega_c = \Omega_1 \cup \Omega_2 \cup (\Sigma \setminus \Gamma_c)$. Obviously, we have $\Sigma \setminus \Gamma_c = \emptyset$, where $\Sigma = \Sigma_0 \setminus \partial\Sigma_0$, $\Sigma_0 = \Gamma_1 \cap \Gamma_2$; hence, in this case, $\Omega_c = \Omega_1 \cup \Omega_2$. On the set Ω_c, we can solve a problem of the form (12) and find the solution $\boldsymbol{u}^\lambda$; after that, on the perturbed set Ω_c^δ, we can solve the problem of finding $\boldsymbol{u}^{\lambda\delta} = (u_1^{\lambda\delta}, u_2^{\lambda\delta})$ and $\sigma^{\lambda\delta} = \{\sigma_{ij}^{\lambda\delta}\}$ $(i,j = 1,2)$ such that

$$-\operatorname{div} \sigma^{\lambda\delta} = \boldsymbol{f} \qquad \text{in} \quad \Omega_c^\delta,$$

$$\sigma^{\lambda\delta} = B^\lambda \varepsilon(\boldsymbol{u}^{\lambda\delta}) \qquad \text{in} \quad \Omega_c^\delta,$$

$$\boldsymbol{u}^{\lambda\delta} = 0 \qquad \text{on} \quad (\partial\Omega_1^\delta \cup \partial\Omega_2^\delta) \setminus \Psi_\delta(\Gamma_c)^\pm,$$

$$[\boldsymbol{u}^{\lambda\delta}] \cdot \boldsymbol{\nu} \geqslant 0, \quad [\sigma_\nu^{\lambda\delta}] = 0, \quad \boldsymbol{\sigma}_\nu^{\lambda\delta} \leqslant 0, \quad \boldsymbol{\sigma}_\tau^{\lambda\delta} = 0, \quad [\boldsymbol{u}^{\lambda\delta}] \cdot \boldsymbol{\nu}\sigma_\nu^{\lambda\delta} = 0 \quad \text{on} \quad \Psi_\delta(\Gamma_c).$$

Here $\boldsymbol{\nu}$ is the internal normal to the boundary $\partial\Omega_1$ determined on Γ_c and $\Omega_i^\delta = \Psi_\delta(\Omega_i)$ $(i = 1,2)$. Note, in this case, we have $\boldsymbol{\nu}^\delta = \Psi_\delta(\boldsymbol{\nu})$. The set Ω_c and the perturbed set Ω_c^δ are not domains because their connectivity is violated. We can find the derivative I^λ of the energy functional in the form (21) and the vector field $\boldsymbol{V}(x) = (\theta(x), 0)$. Hence, formula (21) can be written in the form (27). Integrating Eq. (27) by parts and assuming that $\boldsymbol{f} \equiv 0$ in $S_2 \cap \Omega_c$, we obtain the invariant integral for problem (8)–(11):

$$I^\lambda = \int\limits_{(\partial S_2) \cap \overline{\Omega}_c} \Bigl\{ \frac{1}{2} n_1 \sigma_{ij}^\lambda(\boldsymbol{u}^\lambda) \varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma_{ij}^\lambda(\boldsymbol{u}^\lambda) u_{i,1}^\lambda n_j \Bigr\}$$

[$\boldsymbol{n} = (n_1, n_2)$ is the internal normal to ∂S_2].

By similar considerations, we obtain the invariant integral for the contact problem (2)–(5) from Eq. (26):

$$I^0 = \int\limits_{(\partial S_2) \cap \Omega_1} \Bigl\{ \frac{1}{2} n_1 \sigma_{ij}(\boldsymbol{u}^0) \varepsilon_{ij}(\boldsymbol{u}^0) - \sigma_{ij}(\boldsymbol{u}^0) u_{i,1}^0 n_j \Bigr\}.$$

In this case, $(\partial S_2) \cap \Omega_1$ is a smooth curve connecting the upper and lower edges of the stripe Ω_1.

Example 4. Let the domain Ω_1 have the form of a cone and

$$\Gamma_c = \{(r, \varphi) \mid 0 \leqslant \varphi \leqslant \varphi_0,\ r = q_0(\varphi),\ q_0 > 0,\ q_0 \in C^{0,1}\},$$

$$\Gamma_0 = \{(r, \varphi) \mid \varphi = 0,\ 0 \leqslant r \leqslant q_0(0)\} \cup \{(r, \varphi) \mid \varphi = \varphi_0,\ 0 \leqslant r \leqslant q_0(\varphi_0)\}.$$

Here, (r, φ) are the polar coordinates on the plane. We choose a smooth function θ equal to zero outside some small neighborhood S_1 of the curve Γ_c. Let $\theta = 1$ in the neighborhood S_2 of the curve Γ_c, $S_2 \subset S_1$. The domain Ω_2 is chosen as follows:

$$\Omega_2 = \{(r, \varphi) \mid 0 < \varphi < \varphi_0,\ q_0(\varphi) < r < q_1(\varphi),\ q_1 \in C^{0,1}\}.$$

We determine a (disconnected) set $\Omega_c = \Omega_1 \cup \Omega_2$ and consider a perturbation of the set Ω_c of the form

$$y_1 = x_1(1 + \delta\theta(x)), \qquad y_2 = x_2(1 + \delta\theta(x)), \qquad x \in \Omega_c, \quad y \in \Omega_c^\delta. \tag{32}$$

As previously, we obtain a formula for the derivative of the energy functional in the perturbed problem (15)–(18):

$$I^\lambda = \int\limits_{\Omega_c} \Big\{ \frac{1}{2}\operatorname{div} \boldsymbol{V} \sigma_{ij}^\lambda(\boldsymbol{u}^\lambda)\varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma_{ij}^\lambda(\boldsymbol{u}^\lambda) E_{ij}\Big(\frac{\partial \boldsymbol{V}}{\partial x}; \boldsymbol{u}^\lambda\Big)\Big\} - \int\limits_{\Omega_c} \operatorname{div}(\boldsymbol{V} f_i) u_i^\lambda.$$

We find the vector field for perturbation (32):

$$\boldsymbol{V}(x) = (\theta(x)x_1, \theta(x)x_2).$$

It should be noted that this vector field ensures the equality

$$\int\limits_{S_2\cap\Omega_c} \Big\{ \frac{1}{2}\operatorname{div} \boldsymbol{V} \sigma_{ij}^\lambda(\boldsymbol{u}^\lambda)\varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma_{ij}^\lambda(\boldsymbol{u}^\lambda) E_{ij}\Big(\frac{\partial \boldsymbol{V}}{\partial x}; \boldsymbol{u}^\lambda\Big)\Big\} = 0.$$

Thus, assuming that $\boldsymbol{f} \equiv 0$ in $S_2 \cap \Omega_c$, we obtain

$$I^\lambda = \int\limits_{(S_1\setminus S_2)\cap\Omega_c} \Big\{ \frac{1}{2}\operatorname{div} \boldsymbol{V} \sigma_{ij}^\lambda(\boldsymbol{u}^\lambda)\varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma_{ij}^\lambda(\boldsymbol{u}^\lambda) E_{ij}\Big(\frac{\partial \boldsymbol{V}}{\partial x}; \boldsymbol{u}^\lambda\Big)\Big\} - \int\limits_{(S_1\setminus S_2)\cap\Omega_c} \operatorname{div}(\boldsymbol{V} f_i) u_i^\lambda.$$

Substituting the values of the field $\boldsymbol{V}(x)$ into this equality, we find

$$I^\lambda = \int\limits_{(S_1\setminus S_2)\cap\Omega_c} \Big\{ \frac{1}{2}(\theta_{,l}x_l)\sigma_{ij}^\lambda(\boldsymbol{u}^\lambda)\varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma_{ij}^\lambda(\boldsymbol{u}^\lambda)(u_{i,l}^\lambda x_l)\theta_{,j}\Big\} - \int\limits_{(S_1\setminus S_2)\cap\Omega_c} (x_l\theta f_i)_{,l} u_i^\lambda. \tag{33}$$

We integrate by parts in Eq. (33). Note, after integration by parts, the sum of the integrals over $(S_1 \setminus S_2) \cap \Omega_c$ will be equal to zero; hence, we obtain the invariant integral over $(\partial S_2) \cap \overline{\Omega}_c$ in problem (8)–(11):

$$I^\lambda = \int\limits_{(\partial S_2)\cap\overline{\Omega}_c} \Big\{ \frac{1}{2}(n_l x_l)\sigma_{ij}^\lambda(\boldsymbol{u}^\lambda)\varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma_{ij}^\lambda(\boldsymbol{u}^\lambda)(u_{i,l}^\lambda x_l) n_j\Big\}$$

[$\boldsymbol{n} = (n_1, n_2)$ is the internal normal to the boundary ∂S_2]. The form of this invariant integral differs from those obtained previously.

For the contact problem (2)–(5), the invariant integral has the form

$$I^0 = \int\limits_{(\partial S_2)\cap\Omega_1} \Big\{ \frac{1}{2}(n_l x_l)\sigma_{ij}(\boldsymbol{u}^0)\varepsilon_{ij}(\boldsymbol{u}^0) - \sigma_{ij}(\boldsymbol{u}^0)(u_{i,l}^0 x_l) n_j\Big\}.$$

Three-Dimensional Case. We consider a contact problem in a simply connected bounded domain $\Omega_1 \subset \mathbb{R}^3$ with the Lipschitz boundary Γ_1. Let $\Gamma_c \subset \Gamma_1$ be the contact boundary, i.e., the part of the boundary with the Signorini boundary conditions; $\Gamma_0 = \Gamma_1 \setminus \Gamma_c$, $\operatorname{meas}\Gamma_0 > 0$. For simplicity, we assume that Γ_c as a two-dimensional surface in $\mathbb{R}^3$ can be written as a graph of the function

$$x_3 = \phi(x_1, x_2) \qquad [(x_1, x_2) \in \overline{D}]$$

with a rather smooth function ϕ. Here $D \subset \mathbb{R}^2$ is a simply connected bounded domain with the boundary γ_0 of class $C^{0,1}$, and γ_0, as a curve in $\mathbb{R}^3$, can be written in the form

$$\gamma_0 = \{(r, \varphi, 0) \mid r = g(\varphi),\ \varphi \in [0, 2\pi],\ g(0) = g(2\pi),\ g > 0,\ g \in C^{0,1}\};$$

moreover, there exists $\delta_0 > 0$ such that

$$\{(r, \varphi, 0) \mid g(\varphi) - \delta_0 < r < g(\varphi) + \delta_0\} \subset \Gamma_1. \tag{34}$$

Here (r, φ, ξ) are the cylindrical coordinates in $\mathbb{R}^3$. Condition (34) means that there is a planar segment belonging to the boundary Γ_1 in the vicinity of the edge γ_0 of the contact boundary Γ_c.

The contact problem in the domain Ω_1 is formulated as follows. We have to find functions $\boldsymbol{u}^0 = (u_1^0, u_2^0, u_3^0)$ and $\sigma = \{\sigma_{ij}\}$ $(i, j = 1, 2, 3)$ such that

$$-\operatorname{div}\sigma = \boldsymbol{f} \qquad \text{in} \quad \Omega_1,$$

$$\sigma = C^1\varepsilon(\boldsymbol{u}^0) \qquad \text{in} \quad \Omega_1,$$

$$\boldsymbol{u}^0 = 0 \qquad \text{on} \quad \Gamma_0, \tag{35}$$

$$\boldsymbol{u}^0 \cdot \boldsymbol{\nu} \geqslant 0, \quad \sigma_\nu \leqslant 0, \quad \boldsymbol{\sigma}_\tau = 0, \quad \boldsymbol{u}^0 \cdot \boldsymbol{\nu} \sigma_\nu = 0 \qquad \text{on} \quad \Gamma_c.$$

Here $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$ is the internal normal to $\partial\Omega_1$ on Γ_c, $C^1 = \{c^1_{ijkl}\}$ $(i, j, k, l = 1, 2, 3)$ is the tensor of elasticity moduli, possessing the same properties as that in the two-dimensional case [see (6)], and $\boldsymbol{f} = (f_1, f_2, f_3) \in C^1_{\text{loc}}(\mathbb{R}^3)$. The remaining notation is the same as that used previously.

Problem (35) admits a variational formulation and can be written as a variational inequality. We denote

$$H^1_{\Gamma_0}(\Omega_0) = \{\boldsymbol{v} = (v_1, v_2, v_3) \in H^1(\Omega_c) \mid \boldsymbol{v} = 0 \ \text{ on } \Gamma_0\},$$

$$K = \{\boldsymbol{v} \in H^1_\Gamma(\Omega_c) \mid [\boldsymbol{v}] \cdot \boldsymbol{\nu} \geqslant 0 \text{ a. e. on } \Gamma_c\}.$$

There exists a solution of the variational inequality

$$\boldsymbol{u}^0 \in K, \qquad \int\limits_{\Omega_1} \sigma(\boldsymbol{u}^0)\varepsilon(\boldsymbol{v} - \boldsymbol{u}^0) \geqslant \int\limits_{\Omega_1} \boldsymbol{f}(\boldsymbol{v} - \boldsymbol{u}^0) \quad \forall\, v \in K.$$

As in the two-dimensional case, we construct a bounded domain Ω_2 with the Lipschitz boundary Γ_2. Let $\Omega_c = \Omega_1 \cup \Omega_2 \cup (\Sigma \setminus \Gamma_c)$ and $\Sigma = \Sigma_0 \setminus \partial\Sigma_0$ $(\Sigma_0 = \Gamma_1 \cap \Gamma_2)$. Actually, we assume that $\mathbb{R}^3$ contains a domain divided by a regular surface Σ_0 into two subdomains Ω_1 and Ω_2; $\Gamma_c \subset \Sigma_0$. We denote the outer boundary of the domain Ω_c (i.e., $\partial\Omega_c \setminus \Gamma^\pm_c$) by Γ. The geometry of the domains Ω_1 and Ω_2 is assumed to be such that the cut Γ_c does not reach the outer boundary Γ, i.e., $\Gamma_c \cap \Gamma = \emptyset$. This assumption does not refer to examples 7 and 8 (see below).

We assume that $B^\lambda = \{b^\lambda_{ijkl}\}$ $(\lambda > 0,\ i, j, k, l = 1, 2, 3)$,

$$b^\lambda_{ijkl} = \begin{cases} c^1_{ijkl} & \text{in } \Omega_1, \\ \lambda^{-1} c^2_{ijkl} & \text{in } \Omega_2, \end{cases}$$

where the tensor $C^2 = \{c^2_{ijkl}\}$ possesses the same properties as C^1. In the domain Ω_c with the cut Γ_c, we can find a solution of the family of problems depending on the parameter $\lambda > 0$, namely: for each $\lambda > 0$, we have to find functions $\boldsymbol{u}^\lambda = (u^\lambda_1, u^\lambda_2, u^\lambda_3)$ and $\sigma^\lambda = \{\sigma^\lambda_{ij}\}$ $(i, j = 1, 2, 3)$ such that

$$\begin{gathered} -\operatorname{div} \sigma^\lambda = \boldsymbol{f} \qquad \text{in} \quad \Omega_c, \\ \sigma^\lambda = B^\lambda \varepsilon(\boldsymbol{u}^\lambda) \qquad \text{in} \quad \Omega_c, \\ \boldsymbol{u}^\lambda = 0 \qquad \text{on} \quad \Gamma, \\ [\boldsymbol{u}^\lambda] \cdot \boldsymbol{\nu} \geqslant 0, \quad [\sigma^\lambda_\nu] = 0, \quad \sigma^\lambda_\nu \leqslant 0, \quad \boldsymbol{\sigma}^\lambda_\tau = 0, \quad [\boldsymbol{u}^\lambda] \cdot \boldsymbol{\nu} \sigma^\lambda_\nu = 0 \quad \text{on} \quad \Gamma_c. \end{gathered} \tag{36}$$

Let

$$H^1_\Gamma(\Omega_c) = \{\boldsymbol{v} = (v_1, v_2, v_3) \in H^1(\Omega_c) \mid \boldsymbol{v} = 0 \text{ on } \Gamma\},$$

$$K_c = \{\boldsymbol{v} \in H^1_\Gamma(\Omega_c) \mid [\boldsymbol{v}] \cdot \boldsymbol{\nu} \geqslant 0 \text{ a. e. on } \Gamma_c\}.$$

Then, problem (36) is equivalent to minimization of the functional

$$\Pi_\lambda(\Omega_c; \boldsymbol{v}) = \frac{1}{2} \int\limits_{\Omega_c} \sigma^\lambda(\boldsymbol{v})\varepsilon(\boldsymbol{v}) - \int\limits_{\Omega_c} \boldsymbol{f}\boldsymbol{v}$$

over the set K_c; therefore, the solution $\boldsymbol{u}^\lambda$ of this problem exists and satisfies the variational inequality

$$\boldsymbol{u}^\lambda \in K_c, \quad \int\limits_{\Omega_c} \sigma^\lambda(\boldsymbol{u}^\lambda)\varepsilon(\boldsymbol{v} - \boldsymbol{u}^\lambda) \geqslant \int\limits_{\Omega_c} f(\boldsymbol{v} - \boldsymbol{u}^\lambda) \qquad \forall\, \boldsymbol{v} \in K_c.$$

As a whole, further construction is similar to that performed in the two-dimensional case. We consider the perturbation $y = \Psi_\delta(x)$ of the initial domain in the form

$$y = x + \delta \boldsymbol{V}(x), \qquad x \in \Omega_c, \quad y \in \Omega^\delta_c, \qquad \boldsymbol{V}(x) = (V_1(x), V_2(x), 0).$$

Moreover, we assume that the support of the field $\boldsymbol{V} \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^3)$ does not intersect the boundary Γ. Condition (14) is assumed to be satisfied. Then, we solve a perturbed problem of the form (15)–(18) and find the solution $\boldsymbol{u}^{\lambda\delta}$ and the derivative of the energy functional $\Pi_\lambda(\Omega_c^\delta; \boldsymbol{u}^{\lambda\delta})$ with respect to the parameter δ for $\delta = 0$. Let

$$I^\lambda = \frac{d}{d\delta}\,\Pi_\lambda(\Omega_c^\delta; \boldsymbol{u}^{\lambda\delta})\Big|_{\delta=0}.$$

Similar to (21), we obtain

$$I^\lambda = \int\limits_{\Omega_c} \Big\{\frac{1}{2}\,\operatorname{div}(\boldsymbol{V} b^\lambda_{ijkl})\varepsilon_{kl}(\boldsymbol{u}^\lambda)\varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma^\lambda_{ij}(\boldsymbol{u}^\lambda)E_{ij}\Big(\frac{\partial \boldsymbol{V}}{\partial x}; \boldsymbol{u}^\lambda\Big)\Big\} - \int\limits_{\Omega_c} \operatorname{div}(\boldsymbol{V} f_i)u_i^\lambda, \tag{37}$$

where

$$E_{ij}(\Phi; \boldsymbol{v}) = (v_{i,k}\Phi_{kj} + v_{j,k}\Phi_{ki})/2, \qquad \Phi = \{\Phi_{ij}\}, \qquad i, j, k, l = 1, 2, 3.$$

Note, by virtue of the choice of the vector field $\boldsymbol{V}$ made, there is no need to differentiate the coefficients b^λ_{ijkl} in Eq. (37) with respect to x_3.

Using convergence of the form (23)–(25) again, we obtain a formula for $I^0 = \lim_{\lambda\to 0} I^\lambda$. Indeed,

$$I^0 = \int\limits_{\Omega_1} \Big\{\frac{1}{2}\,\operatorname{div}\boldsymbol{V}\sigma_{ij}(\boldsymbol{u}^0)\varepsilon_{ij}(\boldsymbol{u}^0) - \sigma_{ij}(\boldsymbol{u}^0)E_{ij}\Big(\frac{\partial \boldsymbol{V}}{\partial x}; \boldsymbol{u}^0\Big)\Big\} - \int\limits_{\Omega_1} \operatorname{div}(\boldsymbol{V} f_i)u_i^0. \tag{38}$$

Let us now consider particular choices of the vector field $\boldsymbol{V}(x)$ in formulas (37) and (38), which yield invariant integrals in the three-dimensional case for problems (35) and (36).

Example 5. We choose a smooth function θ with a support in a small neighborhood S_1 of the surface Γ_c. We assume that $\theta = 1$ in the neighborhood S_2 of the surface Γ_c, $S_2 \subset S_1$. The smallness of the neighborhood S_1 means that the edge of the surface $(\partial S_1) \cap \Omega_1$ is a part of the planar segment (34) of the boundary Γ_1. We choose the perturbation of the domain Ω_c in the form

$$y_1 = x_1 + \delta\theta(x_1, x_2, x_3)\cos\alpha, \qquad y_2 = x_2 + \delta\theta(x_1, x_2, x_3)\sin\alpha, \qquad y_3 = x_3,$$

where $(x_1, x_2, x_3) \in \Omega_c$, $(y_1, y_2, y_3) \in \Omega_c^\delta$, and $\alpha \in [0, 2\pi)$ is a fixed number. We denote $p_1 = \cos\alpha$ and $p_2 = \sin\alpha$. In this case, we have $\boldsymbol{V}(x) = (\theta(x)p_1, \theta(x)p_2, 0)$, and formula (37) acquires the form

$$I^\lambda = \int\limits_{\Omega_c} \Big\{\frac{1}{2}\,(\theta_{,l}p_l)\sigma^\lambda_{ij}(\boldsymbol{u}^\lambda)\varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma^\lambda_{ij}(\boldsymbol{u}^\lambda)(u^\lambda_{i,l}p_l)\theta_{,j}\Big\} - \int\limits_{\Omega_c} (\theta f_i)_{,l}p_l u_i^\lambda. \tag{39}$$

Integrating Eq. (39) by parts, we obtain

$$I^\lambda = \int\limits_{(\partial S_2)\cap\overline{\Omega}_c} \Big\{\frac{1}{2}\,(n_l p_l)\sigma^\lambda_{ij}(\boldsymbol{u}^\lambda)\varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma^\lambda_{ij}(\boldsymbol{u}^\lambda)(u^\lambda_{i,l}p_l)n_j\Big\} + \int\limits_{(S_1\backslash S_2)\cap\Omega_c} \theta(\sigma^\lambda_{ij,j} + f_i)(u^\lambda_{i,l}p_l) + \int\limits_{S_2\cap\Omega_c} f_i u^\lambda_{i,l}p_l.$$

Here, $\boldsymbol{n} = (n_1, n_2, n_3)$ is the internal normal to ∂S_2. Assuming that $\boldsymbol{f} \equiv 0$ in $S_2 \cap \Omega_c$, we obtain the invariant integral for problem (36) from the previous relation:

$$I^\lambda = \int\limits_{(\partial S_2)\cap\overline{\Omega}_c} \Big\{\frac{1}{2}\,(n_l p_l)\sigma^\lambda_{ij}(\boldsymbol{u}^\lambda)\varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma^\lambda_{ij}(\boldsymbol{u}^\lambda)(u^\lambda_{i,l}p_l)n_j\Big\} \tag{40}$$

(summation is performed over $i, j = 1, 2, 3$). Similar to formula (38), the invariant integral for the contact problem (35) has the form

$$I^0 = \int\limits_{(\partial S_2)\cap\Omega_1} \Big\{\frac{1}{2}\,(n_l p_l)\sigma_{ij}(\boldsymbol{u}^0)\varepsilon_{ij}(\boldsymbol{u}^0) - \sigma_{ij}(\boldsymbol{u}^0)(u^0_{i,l}p_l)n_j\Big\}. \tag{41}$$

In this case, $(\partial S_2) \cap \Omega_1$ is the "cap"-shaped surface lying in Ω_1 and covering Γ_c.

Example 6. Let $\theta(x)$ be a smooth function equal to zero outside of a small neighborhood S_1 of the curve γ_0, $\theta = 1$ in the neighborhood S_2 of the curve $\gamma_0, S_2 \subset S_1$. For example, S_1 and S_2 are toruses containing γ_0, which are so small that $(\partial S_1) \cap \Gamma_1$ is a part of the planar segment (34). Let us consider a perturbation of the domain Ω_c in the form

$$y_1 = x_1 + \delta\theta(x)p_1, \qquad y_2 = x_2 + \delta\theta(x)p_2, \qquad y_3 = x_3,$$

where $x \in \Omega_c$, $y \in \Omega_c^\delta$, and $p_1^2 + p_2^2 = 1$. We have $\boldsymbol{V}(x) = (\theta(x)p_1, \theta(x)p_2, 0)$, and the formula for I^λ coincides with (39). This case differs from example 5 by the fact that only the neighborhood of the front γ_0 of the crack Γ_c is perturbed.

In this case, the invariant integral for problem (36) for $\boldsymbol{f} \equiv 0$ in $S_2 \cap \Omega_c$ has the form (40).

The same value of the vector field $\boldsymbol{V}(x)$ in (38) yields an invariant integral in problem (35), whose form coincides with (41). In this case, $(\partial S_2) \cap \Omega_1$ is a "cap"-shaped surface lying in Ω_1 and covering the curve γ_0.

The case with only some part of the edge of the boundary Γ_c being perturbed is described by the next example.

Example 7. Let the contact boundary Γ_c be a part of the plane, namely,

$$\Gamma_c = \{(x_1, x_2, 0) \mid 0 \leqslant x_1 \leqslant \phi(x_2),\ \phi(x_2) > 0,\ x_2 \in [-1, 1]\},$$

and there exists $\delta_0 > 0$ such that $\gamma_1 \subset \Gamma_1$, where

$$\gamma_1 = \{(x_1, x_2, 0) \mid 0 \leqslant x_1 \leqslant \phi(x_2) + \delta_0,\ x_2 \in [-1, 1]\}.$$

Here, $\phi(x_2)$ is a rather smooth function. As previously, we consider the domain Ω_2 with a smooth boundary Γ_2 and construct the domain Ω_c. Further, we consider the perturbation of the domain Ω_c with $x \in \Omega_c$ and $y \in \Omega_c^\delta$:

$$y_1 = x_1 + \delta\theta(x), \qquad y_2 = x_2, \qquad y_3 = x_3. \tag{42}$$

The chosen function θ equals zero outside some small three-dimensional neighborhood S_1 of the curve

$$\{(x_1, x_2, 0) \mid x_1 = \phi(x_2),\ x_2 \in [-1, 1]\}. \tag{43}$$

Moreover, $\theta = 1$ in a certain neighborhood S_2 of curve (43), $S_2 \subset S_1$. The smallness of the neighborhood S_1 means that $S_1 \cap \gamma_1$ is a part of the plane. According to (42), we have $\boldsymbol{V}(x) = (\theta(x), 0, 0)$. Then, from formula (37), we obtain

$$I^\lambda = \int\limits_{\Omega_c} \Big\{ \frac{1}{2}\,\theta_{,1}\sigma_{ij}^\lambda(\boldsymbol{u}^\lambda)\varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma_{ij}^\lambda(\boldsymbol{u}^\lambda)u_{i,1}^\lambda\theta_{,j} \Big\} - \int\limits_{\Omega_c} (\theta f_i)_{,1} u_i^\lambda. \tag{44}$$

We integrate Eq. (44) by parts and obtain an invariant integral for problem (36) under the assumption that $\boldsymbol{f} \equiv 0$ in $S_2 \cap \Omega_c$. This integral has the form

$$I^\lambda = \int\limits_{(\partial S_2)\cap\overline{\Omega}_c} \Big\{ \frac{1}{2}\,n_1\sigma_{ij}^\lambda(\boldsymbol{u}^\lambda)\varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma_{ij}^\lambda(\boldsymbol{u}^\lambda)u_{i,1}^\lambda n_j \Big\},$$

where $\boldsymbol{n} = (n_1, n_2, n_3)$ is the internal normal to ∂S_2.

As in other examples, by substituting the chosen field $\boldsymbol{V}(x)$, we use Eq. (38) to find an invariant integral for problem (35):

$$I^0 = \int\limits_{(\partial S_2)\cap\Omega_1} \Big\{ \frac{1}{2}\,n_1\sigma_{ij}(\boldsymbol{u}^0)\varepsilon_{ij}(\boldsymbol{u}^0) - \sigma_{ij}(\boldsymbol{u}^0)u_{i,1}^0 n_j \Big\}.$$

Example 8. Let the domain Ω_1 have the form of a "beam"

$$\Omega_1 = \{(x_1, x_2, x_3) \mid 0 < x_1 < \varphi(x_2, x_3),\ x_2 \in (0, 1),\ x_3 \in (0, 1)\}$$

with a rather smooth function φ such that $\varphi = 1$ for $x_2 = 0, 1$, $x_3 = 0, 1$. We assume that $0 < \varphi(x_2, x_3) < 2$ for $x_2 \in [0, 1]$ and $x_3 \in [0, 1]$. Let the contact boundary Γ_c in the Signorini problem (35) be chosen in the form

$$\Gamma_c = \{(x_1, x_2, x_3) \mid x_1 = \varphi(x_2, x_3),\ x_2 \in [0, 1],\ x_3 \in [0, 1]\}.$$

The domain Ω_2 is also assumed to have the form of a "beam":

$$\Omega_2 = \{(x_1, x_2, x_3) \mid \varphi(x_2, x_3) < x_1 < 2,\ x_2 \in (0, 1),\ x_3 \in (0, 1)\}.$$

We choose a smooth function θ equal to zero outside some small neighborhood S_1 of the surface Γ_c and such that $\theta = 1$ in the neighborhood S_2 of the surface Γ_c, $S_2 \subset S_1$. We consider the perturbation of the set $\Omega_c = \Omega_1 \cup \Omega_2$:

$$y_1 = x_1 + \delta\theta(x), \quad y_2 = x_2, \quad y_3 = x_3, \qquad x \in \Omega_c, \quad y \in \Omega_c^\delta.$$

Note that the set Ω_c in this case is not a domain because the connectivity of Ω_c is violated. We can easily find the vector field $\boldsymbol{V}(x) = (\theta(x), 0, 0)$. Thus, for this vector field $\boldsymbol{V}(x)$, we obtain the following formula from (37):

$$I^\lambda = \int\limits_{\Omega_c} \Big\{ \frac{1}{2}\, \theta_{,1} \sigma^\lambda_{ij}(\boldsymbol{u}^\lambda) \varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma^\lambda_{ij}(\boldsymbol{u}^\lambda) u^\lambda_{i,1} \theta_{,j} \Big\} - \int\limits_{\Omega_c} (\theta f_i)_{,1} u^\lambda_i. \tag{45}$$

Integrating Eq. (45) by parts, we find an invariant integral for problem (36) under the assumption that $\boldsymbol{f} \equiv 0$ in $S_2 \cap \Omega_c$:

$$I^\lambda = \int\limits_{(\partial S_2) \cap \overline{\Omega}_c} \Big\{ \frac{1}{2}\, n_1 \sigma^\lambda_{ij}(\boldsymbol{u}^\lambda) \varepsilon_{ij}(\boldsymbol{u}^\lambda) - \sigma^\lambda_{ij}(\boldsymbol{u}^\lambda) u^\lambda_{i,1} n_j \Big\}$$

[$\boldsymbol{n} = (n_1, n_2, n_3)$ is the internal normal to the boundary ∂S_2].

Similar considerations for $\boldsymbol{f} \equiv 0$ in $S_2 \cap \Omega_c$ yield an invariant integral in problem (35):

$$I^0 = \int\limits_{(\partial S_2) \cap \Omega_1} \Big\{ \frac{1}{2}\, n_1 \sigma_{ij}(\boldsymbol{u}^0) \varepsilon_{ij}(\boldsymbol{u}^0) - \sigma_{ij}(\boldsymbol{u}^0) u^0_{i,1} n_j \Big\}.$$

In particular, we can choose

$$(\partial S_2) \cap \Omega_1 = \{(x_1, x_2, x_3) \mid x_1 = \psi(x_2, x_3),\ x_2 \in (0,1),\ x_3 \in (0,1)\}$$

with a rather smooth function $\psi(x_2, x_3)$ such that

$$0 < \psi(x_2, x_3) < \varphi(x_2, x_3), \qquad x_2 \in (0,1), \quad x_3 \in (0,1).$$

In conclusion, we note that the existence of invariant integrals can also be established in some other cases. In all situations considered above, the value of the invariant integral coincides numerically with the value of the derivative of the energy functional with respect to the perturbation parameter δ for $\delta = 0$. In particular, invariant integrals can be used to approximately find the energy functionals in perturbed problems. As was already noted, the invariant integral I^λ equals the value of the derivative of the energy functional $\Pi_\lambda(\Omega_c^\delta; \boldsymbol{u}^{\lambda\delta})$ with respect to the perturbation parameter δ for $\delta = 0$. Therefore, we can use the formula

$$\Pi_\lambda(\Omega_c^\delta; \boldsymbol{u}^{\lambda\delta}) = \Pi_\lambda(\Omega_c; \boldsymbol{u}^\lambda) + \delta I^\lambda + o(\delta)$$

valid for all $\lambda > 0$. A similar expansion is also valid for $\lambda = 0$, where Ω_c should be replaced by Ω_1.

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